ASYMPTOTIC BEHAVIOUR FOR A CLASS OF SUBHARMONIC FUNCTIONS IN A HALF SPACE *

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ABSTRACT. A class of subharmonic functions are proved to have the growth estimates $u(x)=o(x_n^{1-\frac{\alpha}{p}}|x|^{\frac{\gamma}{p}+\frac{n-1}{q}-n+\frac{\alpha}{p}})$ at infinity in the upper half space of $\mathbf{R}^n$, which generalizes the growth properties of analytic functions and harmonic functions.

1. Introduction and Main Theorem

Let $\mathbf{R}^n(n\geq 3)$ denote the n-dimensional Euclidean space with points $x=(x_1,x_2,\cdots,x_{n-1},x_n)=(x',x_n)$, where $x'\in\mathbf{R}^{n-1}$ and $x_n\in\mathbf{R}$. The boundary and closure of an open Ω of $\mathbf{R}^n$ are denoted by $\partial\Omega$ and $\overline{\Omega}$ respectively. The upper half-space H is the set $H=\{x=(x',x_n)\in\mathbf{R}^n: x_n>0\}$, whose boundary is ∂H . We write $B(x,\rho)$ and $\partial B(x,\rho)$ for the open ball and the sphere of radius ρ centered at x in $\mathbf{R}^n$. We identify $\mathbf{R}^n$ with $\mathbf{R}^{n-1}\times\mathbf{R}$ and $\mathbf{R}^{n-1}$ with $\mathbf{R}^{n-1}\times\{0\}$, with this convention we then have $\partial H=\mathbf{R}^{n-1}$, writing typical points $x,\ y\in\mathbf{R}^n$ as $x=(x',x_n),\ y=(y',y_n)$, where $x'=(x_1,x_2,\cdots,x_{n-1}),\ y'=(y_1,y_2,\cdots y_{n-1})\in\mathbf{R}^{n-1}$ and putting

$$x\cdot y=\sum_{j=1}^{n}x_jy_j=x'\cdot y'+x_ny_n,\quad |x|=\sqrt{x\cdot x},\quad |x'|=\sqrt{x'\cdot x'}.$$

For $x\in\mathbf{R}^n\backslash\{0\}$, let([**10**])

$$E(x)=-r_n|x|^{2-n},$$

where $|x|$ is the Euclidean norm, $r_n=\frac{1}{(n-2)\omega_n}$ and $\omega_n=\frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$ is the surface area of the unit sphere in $\mathbf{R}^n$. We know that E is locally integrable in $\mathbf{R}^n$.

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The Green function $G(x,y)$ for the upper half space H is given by([**10**])

$$G(x,y)=E(x-y)-E(x-y^*) \qquad x,y\in\overline{H},\ x\neq y, \tag{1.1}$$

where * denotes reflection in the boundary plane ∂H just as $y^*=(y_1,y_2,\cdots,y_{n-1},-y_n)$, then we define the Poisson kernel $P(x,y')$ when $x\in H$ and $y'\in\partial H$ by

$$P(x,y')=-\frac{\partial G(x,y)}{\partial y_n}\bigg|_{y_n=0}=\frac{2x_n}{\omega_n|x-(y',0)|^n}. \tag{1.2}$$

The Dirichlet problem of upper half space is to find a function u satisfying

$$u\in C^2(H), \tag{1.3}$$

$$\Delta u=0, x\in H, \tag{1.4}$$

$$\lim_{x\to x'}u(x)=f(x') \text{ nontangentially a.e.} x'\in\partial H, \tag{1.5}$$

where f is a measurable function of $\mathbf{R}^{n-1}$. The Poisson integral of the upper half space is defined by

$$u(x)=P[f](x)=\int_{\mathbf{R}^{n-1}}P(x,y')f(y')dy'. \tag{1.6}$$

As we all know, the Poisson integral $P[f]$ exists if

$$\int_{\mathbf{R}^{n-1}}\frac{|f(y')|}{1+|y'|^n}dy'<\infty.$$

(see [**1,2**] and [**11**])In this paper, we will consider measurable functions f in $\mathbf{R}^{n-1}$ satisfying

$$\int_{\mathbf{R}^{n-1}}\frac{|f(y')|^p}{(1+|y'|)^\gamma}dy'<\infty. \tag{1.7}$$

Siegel-Talvila([**5**]) have proved the following result:

Theorem A Let f be a measurable function in $\mathbf{R}^{n-1}$ satisfying (1.7). Then the harmonic function $v(x)$ defined by (1.6) satisfies (1.3), (1.4), (1.5) and

$$v(x)=o(x_n^{1-n}|x|^{n+m}) \quad \text{as } |x|\to\infty.$$

In order to describe the asymptotic behaviour of subharmonic functions in half-spaces([**8,9**] and [**10**]), we establish the following theorems.

Theorem 1 Let $1\le p<\infty$, $\frac{1}{p}+\frac{1}{q}=1$ and

$$-(n-1)(p-1)<\gamma<(n-1)+p \quad \text{in case } p>1;$$

$$0<\gamma\le n \quad \text{in case } p=1.$$

If f is a measurable function in $\mathbf{R}^{n-1}$ satisfying (1.4) and $v(x)$ is the harmonic function defined by (1.8), then there exists $x_j\in H$, $\rho_j>0$,

such that

$$\sum_{j=1}^{\infty} \frac{\rho_j^{pn-\alpha}}{|x_j|^{pn-\alpha}} < \infty \tag{1.8}$$

holds and

$$v(x) = o(x_n^{1-\frac{\alpha}{p}} |x|^{\frac{\gamma}{p}+\frac{n-1}{q}-n+\frac{\alpha}{p}}) \quad \text{as } |x| \to \infty \tag{1.9}$$

holds in $H - G$. where $G = \bigcup_{j=1}^{\infty} B(x_j, \rho_j)$ and $0 < \alpha \leq n$.

Remark 1 If $\alpha = n$, $p = 1$ and $\gamma = n$, then (1.8) is a finite sum, the set G is the union of finite balls, so (1.9) holds in H. This is just the case $m = 0$ of the result of Siegel-Talvila.

Remark 2 When $\gamma = -(n-1)(p-1)$, $p > 1$, we have

$$v(x) = o(x_n^{1-\frac{\alpha}{p}} (\log |x|)^{\frac{1}{q}} |x|^{\frac{\gamma}{p}+\frac{n-1}{q}-n+\frac{\alpha}{p}}) \quad \text{as } |x| \to \infty \tag{1.10}$$

holds in $H - G$.

Next, we will generalize Theorem 1 to subharmonic functions.

Theorem 2 Let p and γ be as in Theorem 1. If f is a measurable function in $\mathbf{R}^{n-1}$ satisfying (1.7) and μ is a positive Borel measure satisfying

$$\int_H \frac{y_n^p}{(1+|y|)^\gamma} d\mu(y) < \infty \tag{1.10}$$

and

$$\int_H \frac{1}{(1+|y|)^{n-1}} d\mu(y) < \infty.$$

Write the subharmonic function

$$u(x) = v(x) + h(x), \quad x \in H$$

where $v(x)$ is the harmonic function defined by (1.8), $h(x)$ is defined by

$$h(x) = \int_H G(x,y) d\mu(y)$$

and $G(x,y)$ is defined by (1.1). Then there exists $x_j \in H$, $\rho_j > 0$, such that (1.8) holds and

$$u(x) = o(x_n^{1-\frac{\alpha}{p}} |x|^{\frac{\gamma}{p}+\frac{n-1}{q}-n+\frac{\alpha}{p}}) \quad \text{as } |x| \to \infty$$

holds in $H - G$. where $G = \bigcup_{j=1}^{\infty} B(x_j, \rho_j)$ and $0 < \alpha < 2$.

Remark 3 When $\gamma = -(n-1)(p-1)$, $p > 1$, we have

$$u(x) = o(x_n^{1-\frac{\alpha}{p}} (\log |x|)^{\frac{1}{q}} |x|^{\frac{\gamma}{p}+\frac{n-1}{q}-n+\frac{\alpha}{p}}) \quad \text{as } |x| \to \infty \tag{1.10}$$

holds in $H - G$.

2. Proof of Theorem

Let μ be a positive Borel measure in $\mathbf{R}^n$, $\beta \geq 0$, the maximal function $M(d\mu)(x)$ of order β is defined by

$$M(d\mu)(x) = \sup_{0<r<\infty} \frac{\mu(B(x,r))}{r^\beta},$$

then the maximal function $M(d\mu)(x) : \mathbf{R}^n \to [0,\infty)$ is lower semicontinuous, hence measurable. To see this, for any $\lambda > 0$, let $D(\lambda) = \{x \in \mathbf{R}^n : M(d\mu)(x) > \lambda\}$. Fix $x \in D(\lambda)$, then there exists $r > 0$ such that $\mu(B(x,r)) > tr^\beta$ for some $t > \lambda$, and there exists $\delta > 0$ satisfying $(r+\delta)^\beta < \frac{tr^\beta}{\lambda}$. If $|y-x| < \delta$, then $B(y, r+\delta) \supset B(x,r)$, therefore $\mu(B(y, r+\delta)) \geq tr^\beta > \lambda(r+\delta)^\beta$. Thus $B(x,\delta) \subset D(\lambda)$. This proves that $D(\lambda)$ is open for each $\lambda > 0$.

In order to obtain the results, we need these lemmas below:

Lemma 1 Let μ be a positive Borel measure in $\mathbf{R}^n$, $\beta \geq 0$, $\mu(\mathbf{R}^n) < \infty$, for any $\lambda \geq 5^\beta \mu(\mathbf{R}^n)$, set

$$E(\lambda) = \{x \in \mathbf{R}^n : |x| \geq 2, M(d\mu)(x) > \frac{\lambda}{|x|^\beta}\}$$

then there exists $x_j \in E(\lambda)$, $\rho_j > 0$, $j = 1, 2, \cdots$, such that

$$E(\lambda) \subset \bigcup_{j=1}^{\infty} B(x_j, \rho_j) \tag{2.1}$$

and

$$\sum_{j=1}^{\infty} \frac{\rho_j^\beta}{|x_j|^\beta} \leq \frac{3\mu(\mathbf{R}^n)5^\beta}{\lambda}. \tag{2.2}$$

Proof: Let $E_k(\lambda) = \{x \in E(\lambda) : 2^k \leq |x| < 2^{k+1}\}$, then for any $x \in E_k(\lambda)$, there exists $r(x) > 0$, such that $\mu(B(x, r(x))) > \lambda(\frac{r(x)}{|x|})^\beta$, therefore $r(x) \leq 2^{k-1}$. Since $E_k(\lambda)$ can be covered by the union of a family of balls $\{B(x, r(x)) : x \in E_k(\lambda)\}$, by the Vitali Lemma([**6**]), there exists $\Lambda_k \subset E_k(\lambda)$, Λ_k is at most countable, such that $\{B(x, r(x)) : x \in \Lambda_k\}$ are disjoint and

$$E_k(\lambda) \subset \cup_{x\in\Lambda_k} B(x, 5r(x)),$$

so

$$E(\lambda) = \cup_{k=1}^{\infty} E_k(\lambda) \subset \cup_{k=1}^{\infty} \cup_{x\in\Lambda_k} B(x, 5r(x)). \tag{2.3}$$

On the other hand, note that $\cup_{x\in\Lambda_k} B(x, r(x)) \subset \{x : 2^{k-1} \leq |x| < 2^{k+2}\}$, so that

$$\sum_{x\in\Lambda_k} \frac{(5r(x))^\beta}{|x|^\beta} \leq 5^\beta \sum_{x\in\Lambda_k} \frac{\mu(B(x, r(x)))}{\lambda} \leq \frac{5^\beta}{\lambda} \mu\{x : 2^{k-1} \leq |x| < 2^{k+2}\}.$$

Hence we obtain
$$\sum_{k=1}^{\infty}\sum_{x\in\Lambda_k}\frac{(5r(x))^{\beta}}{|x|^{\beta}}\le\sum_{k=1}^{\infty}\frac{5^{\beta}}{\lambda}\mu\{x:2^{k-1}\le|x|<2^{k+2}\}\le\frac{3\mu(\mathbf{R}^n)5^{\beta}}{\lambda}.$$
Rearrange $\{x : x \in \Lambda_k, k = 1, 2, \cdots\}$ and $\{5r(x) : x \in \Lambda_k, k = 1, 2, \cdots\}$, we get $\{x_j\}$ and $\{\rho_j\}$ such that (2.1) and (2.2) hold.

Lemma 2 The kernel $\frac{1}{|x-y|^n}$ has the following estimates:
(1) If $|y|\le\frac{|x|}{2}$, then $\frac{1}{|x-y|^n}\le\frac{2^n}{|x|^n}$;
(2) If $|y|>\frac{|x|}{2}$, then $\frac{1}{|x-y|^n}\le\frac{2^n}{|y|^n}$.

Throughout the proof, A denote various positive constants.
Proof of Theorem 1
We prove only the case $p > 1$; the proof of the case $p = 1$ is similar. Suppose
$$\begin{aligned}
G_1 &= \{y'\in\mathbf{R}^{n-1}:1<|y'|\le\frac{|x|}{2}\},\\
G_2 &= \{y'\in\mathbf{R}^{n-1}:\frac{|x|}{2}<|y'|\le 2|x|\},\\
G_3 &= \{y'\in\mathbf{R}^{n-1}:|y'|>2|x|\},\\
G_4 &= \{y'\in\mathbf{R}^{n-1}:|y'|\le 1\}.
\end{aligned}$$

Define the measure $dm(y')$ by
$$dm(y')=\frac{|f(y')|^p}{(1+|y'|)^{\gamma}}dy'$$
For any $\varepsilon > 0$, there exists $R_\varepsilon > 2$, such that
$$\int_{|y'|\ge R_\varepsilon}dm(y')\le\frac{\varepsilon^p}{5^{pn-\alpha}}.$$
For every Lebesgue measurable set $E\subset\mathbf{R}^{n-1}$, the measure $m^{(\varepsilon)}$ defined by $m^{(\varepsilon)}(E) = m(E\cap\{x'\in\mathbf{R}^{n-1} : |x'|\ge R_\varepsilon\})$ satisfies $m^{(\varepsilon)}(\mathbf{R}^{n-1})\le\frac{\varepsilon^p}{5^{pn-\alpha}}$, write
$$\begin{aligned}
v_1(x) &= \int_{G_1}P(x,y')f(y')dy',\\
v_2(x) &= \int_{G_2}P(x,y')f(y')dy',\\
v_3(x) &= \int_{G_3}P(x,y')f(y')dy',\\
v_4(x) &= \int_{G_4}P(x,y')f(y')dy',
\end{aligned}$$

then

$$v(x) = v_1(x) + v_2(x) + v_3(x) + v_4(x). \tag{2.3}$$

Let $E_1(\lambda) = \{x \in \mathbf{R}^n : |x| \geq 2, \exists t > 0, m^{(\varepsilon)}(B(x,t) \cap \mathbf{R}^{n-1}) > \lambda^p(\frac{t}{|x|})^{pn-\alpha}\}$, therefore, if $|x| \geq 2R_\varepsilon$ and $x \notin E_1(\lambda)$, then we have

$$\forall t > 0,\ m^{(\varepsilon)}(B(x,t) \cap \mathbf{R}^{n-1}) \leq \lambda^p(\frac{t}{|x|})^{pn-\alpha}.$$

First, if $\gamma > -(n-1)(p-1)$, then $\frac{\gamma q}{p} + (n-1) > 0$. For $r > 1$, we have

$$v_1(x) = \int_{G_1 \cap B(0,r)} P(x,y')f(y')dy' + \int_{G_1 - B(0,r)} P(x,y')f(y')dy' = v_{11}(x) + v_{12}(x).$$

If $|x| > 2r$, then we obtain by Lemma 2 (1) and Holder's inequality

$$\begin{aligned} |v_{11}(x)| &\leq \int_{B(0,r)-B(0,1)} \frac{2x_n}{\omega_n}\frac{2^n}{|x|^n}|f(y')|dy' \\ &\leq \frac{2^{n+1}}{\omega_n}\frac{x_n}{|x|^n}\Big(\int_{B(0,r)-B(0,1)} \frac{|f(y')|^p}{|y'|^\gamma}dy'\Big)^{1/p}\Big(\int_{B(0,r)-B(0,1)} |y'|^{\frac{\gamma q}{p}}dy'\Big)^{1/q}, \end{aligned}$$

since

$$\int_{B(0,r)-B(0,1)} |y'|^{\frac{\gamma q}{p}}dy' \leq \omega_{n-1}\frac{1}{\frac{\gamma q}{p}+n-1}r^{\frac{\gamma q}{p}+n-1},$$

so that

$$v_{11}(x) = o(x_n|x|^{\frac{\gamma}{p}+\frac{n-1}{q}-n}) \quad \text{as } |x| \to \infty. \tag{1.9}$$

Moreover, we have similarly

$$\begin{aligned} |v_{12}(x)| &\leq \frac{2^{n+1}}{\omega_n}\frac{x_n}{|x|^n}\Big(\int_{G_1-B(0,r)} \frac{|f(y')|^p}{|y'|^\gamma}dy'\Big)^{1/p}\Big(\int_{G_1-B(0,r)} |y'|^{\frac{\gamma q}{p}}dy'\Big)^{1/q} \\ &\leq Ax_n|x|^{\frac{\gamma}{p}+\frac{n-1}{q}-n}\Big(\int_{G_1-B(0,r)} \frac{|f(y')|^p}{|y'|^\gamma}dy'\Big)^{1/p}, \end{aligned}$$

which implies by artitrariness of r that

$$v_{12}(x) = o(x_n|x|^{\frac{\gamma}{p}+\frac{n-1}{q}-n}) \quad \text{as } |x| \to \infty. \tag{1.9}$$

If $\gamma > -(n-1)(p-1)$, then $\frac{\gamma q}{p} + (n-1) > 0$, so that we obtain by Holder's inequality

$$\begin{aligned} |v_2(x)| &\leq \frac{2x_n}{\omega_n}\Big(\int_{G_2} \frac{|f(y')|^p}{|x-(y',0)|^{pn}|y'|^\gamma}dy'\Big)^{1/p}\Big(\int_{G_2} |y'|^{\frac{\gamma q}{p}}dy'\Big)^{1/q} \\ &\leq Ax_n|x|^{\frac{\gamma}{p}+\frac{n-1}{q}}\Big(\int_{G_2} \frac{|f(y')|^p}{|x-(y',0)|^{pn}|y'|^\gamma}dy'\Big)^{1/p}, \end{aligned}$$

since

$$\begin{aligned}\int_{G_2}\frac{|f(y')|^p}{|x-(y',0)|^{pn}|y'|^\gamma}dy' &\leq \int_{\frac{x_n}{2}}^{3|x|}\frac{2^\gamma+1}{t^{pn}}dm_x^{(\varepsilon)}(t)\\ &\leq \frac{\varepsilon^p}{|x|^{pn}}(2^\gamma+1)\left(\frac{1}{3^\alpha}+\frac{pn}{\alpha}\right)\frac{|x|^\alpha}{x_n^\alpha},\end{aligned}$$

where $m_x^{(\varepsilon)}(t)=\int_{|y'-x|\leq t}dm^{(\varepsilon)}(y')$.
Hence we have

$$v_2(x)=o(x_n^{1-\frac{\alpha}{p}}|x|^{\frac{\gamma}{p}+\frac{n-1}{q}-n+\frac{\alpha}{p}}) \quad \text{as } |x|\to\infty.$$

If $\gamma<(n-1)+p$, then $(\frac{\gamma}{p}-n)q+(n-1)<0$, so that we obtain by Lemma 2 (2) and Holder's inequality

$$\begin{aligned}|v_3(x)| &\leq \int_{G_3}\frac{2x_n}{\omega_n}\frac{2^n}{|y'|^n}|f(y')|dy'\\ &\leq \frac{2^{n+1}}{\omega_n}x_n\left(\int_{G_3}\frac{|f(y')|^p}{|y'|^\gamma}dy'\right)^{1/p}\left(\int_{G_3}|y'|^{(\frac{\gamma}{p}-n)q}dy'\right)^{1/q}\\ &\leq Ax_n|x|^{\frac{\gamma}{p}+\frac{n-1}{q}-n}\left(\int_{G_3}\frac{|f(y')|^p}{|y'|^\gamma}dy'\right)^{1/p},\end{aligned}$$

so that

$$v_3(x)=o(x_n|x|^{\frac{\gamma}{p}+\frac{n-1}{q}-n}) \quad \text{as } |x|\to\infty. \tag{1.9}$$

Finally, by Lemma 2 (1), we obtain

$$|v_4(x)|\leq\frac{2^{n+1}}{\omega_n}\frac{x_n}{|x|^n}\int_{G_4}|f(y')|dy',$$

so that we have by $\gamma>-(n-1)(p-1)$

$$v_4(x)=o(x_n|x|^{\frac{\gamma}{p}+\frac{n-1}{q}-n}) \quad \text{as } |x|\to\infty. \tag{1.9}$$

Thus, by collecting (2.5), (2.6), (2.7), (2.8), (2.9), (2.10)and (2.11), there exists a positive constant A independent of ε, such that if $|x|\geq 2R_\varepsilon$ and $x\notin E_1(\varepsilon)$, we have

$$|v(x)|\leq A\varepsilon x_n^{1-\frac{\alpha}{p}}|x|^{\frac{\gamma}{p}+\frac{n-1}{q}-n+\frac{\alpha}{p}}.$$

Let μ_ε be a measure in $\mathbf{R}^n$ defined by $\mu_\varepsilon(E)=m^{(\varepsilon)}(E\cap\mathbf{R}^{n-1})$ for every measurable set E in $\mathbf{R}^n$.Take $\varepsilon=\varepsilon_p=\frac{1}{2^{p+2}}, p=1,2,3,\cdots$, then there exists a sequence $\{R_p\}$: $1=R_0<R_1<R_2<\cdots$ such that

$$\mu_{\varepsilon_p}(\mathbf{R}^n)=\int_{|y'|\geq R_p}dm(y')<\frac{\varepsilon_p^p}{5^{pn-\alpha}}.$$

Take $\lambda = 3 \cdot 5^{pn-\alpha} \cdot 2^p \mu_{\varepsilon_p}(\mathbf{R}^n)$ in Lemma 1, then there exists $x_{j,p}$ and $\rho_{j,p}$, where $R_{p-1} \le |x_{j,p}| < R_p$, such that

$$\sum_{j=1}^{\infty}(\frac{\rho_{j,p}}{|x_{j,p}|})^{pn-\alpha} \le \frac{1}{2^p}.$$

if $R_{p-1} \le |x| < R_p$ and $x \notin G_p = \cup_{j=1}^{\infty} B(x_{j,p}, \rho_{j,p})$, we have

$$|v(x)| \le A\varepsilon_p x_n^{1-\frac{\alpha}{p}} |x|^{\frac{\gamma}{p}+\frac{n-1}{q}-n+\frac{\alpha}{p}},$$

Thereby

$$\sum_{p=1}^{\infty}\sum_{j=1}^{\infty}(\frac{\rho_{j,p}}{|x_{j,p}|})^{pn-\alpha} \le \sum_{p=1}^{\infty}\frac{1}{2^p} = 1 < \infty.$$

Set $G = \cup_{p=1}^{\infty} G_p$, thus Theorem 1 holds.

Proof of Theorem 2

We prove only the case $p > 1$; the remaining case $p = 1$ can be proved similarly. Suppose

$$\begin{aligned} F_1 &= \{y \in H : 1 < |y| \le \frac{|x|}{2}\}, \\ F_2 &= \{y \in H : \frac{|x|}{2} < |y| \le 2|x|\}, \\ F_3 &= \{y \in H : |y| > 2|x|\}, \\ F_4 &= \{y \in H : |y| \le 1\}. \end{aligned}$$

Define the measure $dn(y)$ by

$$dn(y) = \frac{y_n^p}{(1+|y'|)^{\gamma}} d\mu(y)$$

For any $\varepsilon > 0$, there exists $R_\varepsilon > 2$, such that

$$\int_{|y|\ge R_\varepsilon} dn(y) < \frac{\varepsilon^p}{5^{pn-\alpha}}.$$

For every Lebesgue measurable set $E \subset \mathbf{R}^n$, the measure $n^{(\varepsilon)}$ defined by $n^{(\varepsilon)}(E) = n(E \cap \{y \in H : |y| \ge R_\varepsilon\})$ satisfies $n^{(\varepsilon)}(H) \le \frac{\varepsilon^p}{5^{pn-\alpha}}$, write

$$\begin{aligned} h_1(x) &= \int_{F_1} G(x,y) d\mu(y), \\ h_2(x) &= \int_{F_2} G(x,y) d\mu(y), \\ h_3(x) &= \int_{F_3} G(x,y) d\mu(y), \\ h_4(x) &= \int_{F_4} G(x,y) d\mu(y) \end{aligned}$$

then

$$h(x) = h_1(x) + h_2(x) + h_3(x) + h_4(x). \tag{2.10}$$

Let $E_2(\lambda) = \{x \in \mathbf{R}^n : |x| \geq 2, \exists t > 0, n^{(\varepsilon)}(B(x,t) \cap H) > \lambda^p(\frac{t}{|x|})^{pn-\alpha}\}$, therefore, if $|x| \geq 2R_\varepsilon$ and $x \notin E_2(\lambda)$, then we have

$$\forall t > 0,\ n^{(\varepsilon)}(B(x,t) \cap H) \leq \lambda^p(\frac{t}{|x|})^{pn-\alpha}.$$

First, note that

$$|G(x,y)| = |E(x-y) - E(x-y^*)| \leq \frac{2x_n y_n}{\omega_n |x-y|^n}. \tag{2.11}$$

If $\gamma > -(n-1)(p-1)$, then $\frac{\gamma q}{p} + (n-1) > 0$. For $r > 1$, we have

$$h_1(x) = \int_{F_1 \cap B(0,r)} -G(x,y)d\mu(y) + \int_{F_1 - B(0,r)} -G(x,y)d\mu(y) = h_{11}(x) + h_{12}(x)$$

If $|x| > 2r$, then we obtain by Lemma 2 (1), (2.11) and Holder's inequality

$$\begin{aligned} |h_{11}(x)| &\leq \int_{B(0,r)-B(0,1)} \frac{2x_n y_n}{\omega_n |x-y|^n} d\mu(y) \\ &\leq \int_{B(0,r)-B(0,1)} \frac{2x_n y_n}{\omega_n} \frac{2^n}{|x|^n} d\mu(y) \\ &\leq \frac{2^{n+1}}{\omega_n} \frac{x_n}{|x|^n} \left(\int_{B(0,r)-B(0,1)} \frac{y_n^p}{|y|^\gamma} d\mu(y) \right)^{1/p} \left(\int_{B(0,r)-B(0,1)} |y|^{\frac{\gamma q}{p}} d\mu(y) \right)^{1/q}, \end{aligned}$$

since

$$\int_{B(0,r)-B(0,1)} |y|^{\frac{\gamma q}{p}} d\mu(y) \leq 2^{n-1} r^{\frac{\gamma q}{p}+n-1} \int_H \frac{1}{(1+|y|)^{n-1}} d\mu(y),$$

$$\begin{aligned} \int_{B(0,r)-B(0,1)} |y|^{\frac{\gamma q}{p}} d\mu(y) &= \int_{B(0,r)-B(0,1)} |y|^{\frac{\gamma q}{p}+n-1} \frac{1}{|y|^{n-1}} d\mu(y) \\ &\leq 2^{n-1} \int_H \frac{1}{(1+|y|)^{n-1}} d\mu(y) r^{\frac{\gamma q}{p}+n-1}, \end{aligned}$$

so that

$$h_{11}(x) = o(x_n |x|^{\frac{\gamma}{p}+\frac{n-1}{q}-n}) \quad \text{as } |x| \to \infty. \tag{1.9}$$

Moreover, we have similarly

$$\begin{aligned} |h_{12}(x)| &\leq \frac{2^{n+1}}{\omega_n} \frac{x_n}{|x|^n} \left(\int_{F_1 - B(0,r)} \frac{y_n^p}{|y|^\gamma} d\mu(y) \right)^{1/p} \left(\int_{F_1 - B(0,r)} |y|^{\frac{\gamma q}{p}} d\mu(y) \right)^{1/q} \\ &\leq A x_n |x|^{\frac{\gamma}{p}+\frac{n-1}{q}-n} \left(\int_{F_1 - B(0,r)} \frac{y_n^p}{|y|^\gamma} d\mu(y) \right)^{1/p}, \end{aligned}$$

which implies by artitrariness of r that

$$h_{12}(x) = o(x_n|x|^{\frac{\gamma}{p}+\frac{n-1}{q}-n}) \quad \text{as } |x| \to \infty. \tag{1.9}$$

If $\gamma > -(n-1)(p-1)$, then $\frac{\gamma q}{p} + (n-1) > 0$, so that we obtain by Holder's inequality

$$\begin{aligned}
|h_2(x)| &\leq \left(\int_{F_2} \frac{|G(x,y)|^p}{|y|^\gamma} d\mu(y)\right)^{1/p} \left(\int_{F_2} |y|^{\frac{\gamma q}{p}} d\mu(y)\right)^{1/q} \\
&\leq \left(\int_{F_2} \frac{|G(x,y)|^p}{y_n^p} (2^\gamma + 1) dn(y)\right)^{1/p} \left(\int_{F_2} |y|^{\frac{\gamma q}{p}} d\mu(y)\right)^{1/q} \\
&\leq A|x|^{\frac{\gamma}{p}+\frac{n-1}{q}} \left(\int_{F_2} \frac{|G(x,y)|^p}{y_n^p} dn(y)\right)^{1/p},
\end{aligned}$$

since

$$\begin{aligned}
\int_{F_2} \frac{|G(x,y)|^p}{y_n^p} dn(y) &\leq \int_{|y-x|\leq 3|x|} \frac{|G(x,y)|^p}{y_n^p} dn^{(\varepsilon)}(y) \\
&= \int_{|y-x|\leq \frac{x_n}{2}} \frac{|G(x,y)|^p}{y_n^p} dn^{(\varepsilon)}(y) + \int_{\frac{x_n}{2}<|y-x|\leq 3|x|} \frac{|G(x,y)|^p}{y_n^p} dn^{(\varepsilon)}(y) \\
&= h_{21}(x) + h_{22}(x),
\end{aligned}$$

so that

$$\begin{aligned}
h_{21}(x) &\leq \int_{|y-x|\leq \frac{x_n}{2}} \left(\frac{2}{(n-2)\omega_n x_n |x-y|^{(n-2)}}\right)^p dn^{(\varepsilon)}(y) \\
&= \left(\frac{2}{(n-2)\omega_n x_n}\right)^p \int_0^{\frac{x_n}{2}} \frac{1}{t^{p(n-2)}} dn_x^{(\varepsilon)}(t) \\
&\leq \left(\frac{2}{(n-2)\omega_n}\right)^p \frac{np-\alpha}{(2p-\alpha)2^{2p-\alpha}} \varepsilon^p \frac{x_n^{p-\alpha}}{|x|^{np-\alpha}}.
\end{aligned}$$

Moreover, we have by (2.11)

$$\begin{aligned}
h_{22}(x) &\leq \int_{\frac{x_n}{2}<|y-x|\leq 3|x|} \left(\frac{2x_n}{(n-2)\omega_n |x-y|^n}\right)^p dn^{(\varepsilon)}(y) \\
&= \left(\frac{2x_n}{\omega_n}\right)^p \int_{\frac{x_n}{2}}^{3|x|} \frac{1}{t^{pn}} dn_x^{(\varepsilon)}(t) \\
&\leq \left(\frac{2}{\omega_n}\right)^p \left(\frac{1}{3^\alpha} + \frac{np2^\alpha}{\alpha}\right) \varepsilon^p \frac{x_n^{p-\alpha}}{|x|^{np-\alpha}},
\end{aligned}$$

where $n_x^{(\varepsilon)}(t) = \int_{|y-x|\leq t} dn^{(\varepsilon)}(y)$.

Hence we have

$$h_2(x) = o(x_n^{1-\frac{\alpha}{p}} |x|^{\frac{\gamma}{p}+\frac{n-1}{q}-n+\frac{\alpha}{p}}) \quad \text{as } |x| \to \infty.$$

If $\gamma < (n-1)+p$, then $(\frac{\gamma}{p}-n)q+(n-1)<0$, so that we obtain by Lemma 2 (2), (2.11) and Holder's inequality

$$\begin{aligned}
|h_3(x)| &\leq \int_{F_3} \frac{2x_n y_n}{\omega_n |x-y|^n} d\mu(y) \\
&\leq \int_{F_3} \frac{2x_n y_n}{\omega_n} \frac{2^n}{|y|^n} d\mu(y) \\
&\leq \frac{2^{n+1}}{\omega_n} x_n \left(\int_{F_3} \frac{y_n^p}{|y|^\gamma} d\mu(y) \right)^{1/p} \left(\int_{F_3} |y|^{(\frac{\gamma}{p}-n)q} d\mu(y) \right)^{1/q} \\
&\leq A x_n |x|^{\frac{\gamma}{p}+\frac{n-1}{q}-n} \left(\int_{F_3} \frac{y_n^p}{|y|^\gamma} d\mu(y) \right)^{1/p},
\end{aligned}$$

so that

$$h_3(x) = o(x_n |x|^{\frac{\gamma}{p}+\frac{n-1}{q}-n}) \quad \text{as } |x| \to \infty. \tag{1.9}$$

Finally, by Lemma 2 (1) and (2.11), we obtain

$$|h_4(x)| \leq \int_{F_4} \frac{2x_n y_n}{\omega_n |x-y|^n} d\mu(y) \leq \frac{2^{n+1}}{\omega_n} \frac{x_n}{|x|^n} \int_{F_4} y_n d\mu(y),$$

so that we have by $\gamma > -(n-1)(p-1)$

$$h_4(x) = o(x_n |x|^{\frac{\gamma}{p}+\frac{n-1}{q}-n}) \quad \text{as } |x| \to \infty. \tag{1.9}$$

Thus, by collecting (2.12), (2.13), (2.15), (2.16), (2.17), (2.18), (2.19) and (2.20), there exists a positive constant A independent of ε, such that if $|x| \geq 2R_\varepsilon$ and $x \notin E_2(\varepsilon)$, we have

$$|h(x)| \leq A\varepsilon x_n^{1-\frac{\alpha}{p}} |x|^{\frac{\gamma}{p}+\frac{n-1}{q}-n+\frac{\alpha}{p}}.$$

Similarly, if $x \notin G$, we have

$$h(x) = o(x_n^{1-\frac{\alpha}{p}} |x|^{\frac{\gamma}{p}+\frac{n-1}{q}-n+\frac{\alpha}{p}}) \quad \text{as } |x| \to \infty. \tag{2.21}$$

by (1.11) and (2.21), we obtain

$$u(x) = v(x) + h(x) = o(x_n^{1-\frac{\alpha}{p}} |x|^{\frac{\gamma}{p}+\frac{n-1}{q}-n+\frac{\alpha}{p}}) \quad \text{as } |x| \to \infty$$

hold in $H - G$, thus we complete the proof of Theorem 2.

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